1. Philosophical questions & mathematical answers

Historically, math has been an important test case for settling several important philosophical debates/questions:

1.1. Rationalism v. empiricism

**Rationalism**: the thesis that all knowledge (ethical, empirical, etc.) is the derivation of conclusions from self-evident principles (axioms).
- Derivation from self-evident principles = “pure reason”
- Takes math as the paradigmatic form of knowledge

**Empiricism**: the thesis that all knowledge is acquired through the five senses.
- The output of five senses = “experience.”
- Seems to have trouble accommodating mathematical knowledge.

1.2. Is a priori knowledge possible?

**A priori knowledge**: knowledge that is not acquired through experience.

**A posteriori knowledge**: knowledge that is acquired through experience.

1.3. Other important philosophical issues

**Logic**: When is one statement a good reason to affirm another? Mathematical proof is a paradigm case of correct reasoning.

**Philosophy of language**: How do words/sentences (a) convey meaning, and (b) map onto the world?
- Mathematical language is among our most precise languages, so it provides a cleaner example than a natural language (e.g. English.)

**Normativity**: What does it mean for something to be correct/incorrect, wrong/right, etc.?
- Mathematical logic has very clear rules of when it is correct and incorrect (to draw certain conclusions), so it is simpler than other normative systems (e.g. ethics.)

2. What questions does philosophy of mathematics seek to answer?

**Big picture**: How does mathematics function in our broader intellectual lives?

**Ontology** (the study of what exists): What is the subject-matter of mathematics?

**Philosophy of science**: Why do so many sciences use mathematics to understand the world? How do the subject matters of math and of science relate to each other so as to produce this understanding?

**Epistemology** (the study of knowledge): How do we come to know mathematics?

**Philosophy of language**: How is mathematical language to be understood?

3. What comes first: math or philosophy?

To what extent can we expect philosophy of math to determine or even suggest the proper practice of math? Conversely, to what extent can we expect the autonomous practice of math to determine the correct philosophy of math?

3.1. Philosophy first

The philosophy-first principle: philosophy must first define what various mathematical concepts are, and then derive principles of good mathematical practice from these definitions.

3.1.1. Plato's Argument (pp. 7-8)

P1. If mathematicians should be interpreted literally, then mathematical objects are created and changed.

P2. If mathematical objects are created, then they are not eternal.

P3. Mathematical objects are eternal and unchanging.

C. Therefore, mathematicians should not be interpreted literally (from P1-P3)
What are the methods of philosophy of mathematics?

3.1.2. Brouwer’s Intuitionism / Dummett’s Verificationism

P1. If a statement can be neither proven nor disproven, then it is neither true nor false.
P2. Some statements can be neither proven nor disproven.
C1. \(\therefore\) Some statements are neither true nor false. (from P1, P2)
P3. If the law of excluded middle is true\(^1\), then every statement is either true or false.
C2. \(\therefore\) The law of excluded middle is not true. (from C1, P3)
P4. Most mathematicians assume that the law of excluded middle is true.
C3. Most mathematicians assume something false. (from C2, P4)

3.2. Philosophy-last approach

The philosophy-last-if-at-all principle: if philosophy of mathematics has any use at all, it is to provide a coherent account of mathematical practice. We start with the success of mathematics, and see which assumptions have been essential to that success. Those essential assumptions \(\approx\) the correct philosophy of mathematics.

3.2.1. Rebuttal to Brouwer

P1. If analysis, algebra, topology, etc. are successful enterprises, then the law of excluded middle is true.
P2. Analysis, algebra, topology, etc. are successful enterprises.
C1. \(\therefore\) The law of excluded middle is true. (from P1, P2)
P3. If the law of excluded middle is true, then every statement is either true or false.
C2. Every statement is either true or false. (from C1, P3)
C3. \(\therefore\) Every statement is either true or false, even those statements that can be neither proven nor disproven. (from C2)

3.3. A false dilemma? Philosophy-in-between

“the correct way to do mathematics is not a direct consequence of the true philosophy, nor is the correct philosophy of mathematics an immediate consequence of mathematics as practiced.” (15)

3.3.1. Against philosophy-last

P1. All philosophy-of-math questions are philosophical questions.
P2. Philosophical questions are not mathematical questions. (There’s something funny about this premise. Can you say what it is?)
C1. \(\therefore\) Philosophy-of-math questions are not mathematical questions. (from P1, P2)
P3. If philosophy-of-math questions are not mathematical questions, then the philosophy-last-if-at-all principle is false.
C2. \(\therefore\) The philosophy-last-if-at-all principle is false. (from C1, P3)

3.3.2. Against philosophy-first

P1. The best answer to philosophy-of-math questions must cohere with mathematical practice.
P2. If the best answer to philosophy-of-math questions must cohere with mathematical practice, then the philosophy-first principle is false.
C1. \(\therefore\) The philosophy-first principle is false. (from P1, P2)

3.3.3. Combining the two arguments

Philosophy of math has its own set of questions, which are not the same as mathematical questions, yet the answers to those questions should cohere with mathematical practice.

\(^1\) The law of excluded middle is only one such example. Another is the inference from 
\[\neg\forall xPx\] to \[\exists x\neg Px\].
What are the methods of philosophy of mathematics?

4. Does science come before math and philosophy?

4.1. What is Naturalism?

**Naturalism**: the thesis that answers to all philosophical questions must cohere with our best scientific theories.

- **Naturalized epistemology**: Philosophical question, “What is knowledge?” must cohere with cognitive psychology.
- **Naturalized ontology**: Philosophical question, “What exists?” must cohere with physics.

Rejects philosophy-first principle, but *also* adds a “science-first principle.”

4.2. Quine's Naturalism

W.V. Quine was the leading proponent of naturalism and also endorsed holism.

4.2.1. Holism

**Holism**: propositions are justified not by some privileged set of axioms/data, but by their overall fit within a larger network or web of propositions that are interconnected by inferential and explanatory relationships.

Ramifications for philosophy of mathematics: philosophy and math are mutually constraining. We’re trying to get our best fit between certain philosophical principles and mathematical practice.

4.2.2. Quine on math

**P1.** For all statements \( p \), if \( p \) plays some inferential role (however indirect) in the parts of science that bear on sensory experience, we have good reason to believe that \( p \) is true; otherwise, we have good reason not to believe that \( p \) is true.

**P2.** Some mathematical statements play some inferential role in the parts of science that bear on sensory experience; others do not.

**C.** ∴ We have good reason to believe that some mathematical statements are true; we have good reason not to believe that other mathematical statements are true. (from P1, P2)

4.3. Maddy's critique of Quine

Maddy calls herself a “naturalist,” but she is closer to a philosophy-last position than Quine’s naturalism. While Quine appeals to scientific practice, Maddy appeals to mathematical practice.

4.3.1. Autonomy argument

**P1.** If there is no overarching theory that covers all branches of science and mathematics, then science and mathematics need not have the same goal.

**P2.** There is no overarching theory that covers all branches of natural science and mathematics.

**C1.** ∴ Science and mathematics need not have the same goal (from P1, P2)

**P3.** If science and mathematics do not have the same goal, then there can be mathematical statements \( p \) such that \( p \) plays no inferential role in the parts of science that bear on sensory experience, yet we still have good reason to believe that \( p \) is true.

**C2.** ∴ There can be mathematical statements \( p \) such that \( p \) plays no inferential role in the parts of science that bear on sensory experience, yet we still have good reason to believe that \( p \) is true. (from C1, P3)

4.3.2. Long-run argument

**P1.** If something frequently aids science in explaining and predicting sensory experience, then it should be accepted.

**P2.** Mathematical ideas with no inferential role in the parts of science that bear directly on sensory experience (in the short-run) have frequently aided science in explaining and predicting sensory experience (in the long-run).

**C.** ∴ So mathematical ideas with no (short-term) inferential role in the parts of science that bear directly on sensory experience should be accepted. (from P1, P2)