1. Historical background

16th century: two largest philosophical movements are rationalism and empiricism.

1.1. Commonalities

Neither had very well developed philosophies of mathematics.

Both (tacitly) accepted truth-value realism about mathematics.

Both were ontological realists about mathematics, holding that mathematics is about physical magnitudes and extended objects that are encountered empirically.

Both granted that once we have basic mathematical concepts (axioms), there is no need to gain further mathematical knowledge through the senses. (Hume on “relations of ideas” vs. “matters of fact”)

1.2. Differences

Nativism: Rationalists believed that some ideas were innate; empiricists did not.

Epistemology: Rationalists thought we could gain a priori knowledge through “rational insight”; empiricists deny this.

- Rationalist view nicely accounts for the necessity of mathematical statements: we know that \( p \) is necessary only if we have rational insight into \( p \).
- Empiricist proposals leave something to be desired:
  - First proposal: Mathematical statements are (mere) definitions. Captures necessity of statements, but “leaves math without substance.” (Shapiro 2000: 76)
  - Second proposal: A statement \( p \) is necessary if and only if we cannot imagine/conceive of \( \neg p \). Seems to capture a very weak notion of necessity.

Applicability: Empiricists held that mathematical ideas are abstracted from observation and that mathematicians study the relations of ideas; rationalists favor nativism over abstractionism.

- Rationalist proposal: mathematical ideas are innate, eternal, and causally inert, so it’s unclear how they apply to empirical things that are not ideas, changing, and causally interactive.

2. Kant

Sought to find a happy medium between empiricism and rationalism; especially to preserve the a priori nature of mathematical knowledge, without making its applicability to science mysterious.

Key idea is a finer-grained account of the a priori than his predecessors

2.1. Key concepts

2.1.1. Analytic vs. synthetic statements

<table>
<thead>
<tr>
<th><strong>Analytic statements</strong></th>
<th><strong>Synthetic statements based on pure intuitions</strong></th>
<th><strong>Synthetic statements based on empirical intuitions</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>A universal proposition (“All S's are P”) is analytic iff the subject concept (S) “contains” the predicate concept.</td>
<td>Not analytic</td>
<td>Not analytic</td>
</tr>
<tr>
<td>Based on concepts (see below)</td>
<td>Based on pure intuitions (see 2.1.2 below)</td>
<td>Based on empirical intuitions (i.e. sensory experiences)</td>
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<tr>
<td>( A ) priori: if ( A ) is an analytic truth and ( S ) grasps all of the concepts expressed in ( A ), then ( S ) is in a position to determine their parts and thus the truth of ( A ). (No sensory experience required.)</td>
<td>A priori (see 2.2.2 below)</td>
<td>A posteriori</td>
</tr>
<tr>
<td>Grasped through conceptual analysis</td>
<td>Grasped through mental constructions (see below)</td>
<td>Grasped through senses</td>
</tr>
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### 2.1.2. Concepts vs. intuitions

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Intuitions</th>
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<tbody>
<tr>
<td>Universal (All F’s are G’s)</td>
<td>Singular (Here is a)</td>
</tr>
<tr>
<td>No existential import: simply grasping what something <em>means</em> doesn’t entail that it <em>exists.</em></td>
<td>Existential import: To have an intuition of <em>a</em> is to be committed to <em>a’s</em> existence.</td>
</tr>
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</table>

Intuitions come in two flavors:

- *Empirical intuitions* are sensory perceptions of objects.
- *Pure intuitions* are forms of all of our possible empirical intuitions. For Kant, there are only two kinds of pure intuition: the intuition of space and the intuition of time.

#### 2.1.3. The *a priori*

Prior to Kant, it was assumed that only *analytic* statements could be *a priori*; and that all *synthetic* statements are *a posteriori*.

By contrast, Kant claims that some *synthetic* statements are also *a priori*. These are statements that are presuppositions of any possible sensory experience, but are neither acquired nor justified empirically.

The paradigmatic cases of synthetic a priori statements are those of mathematics. (However, Kant had a number of others! For instance, that *every event has a cause*. More on this tomorrow.)

In particular, mathematical knowledge is *synthetic* because it’s based in intuition, and an *a priori* because it’s based in *pure* intuition, not *empirical* intuition.

#### 2.2. Kant’s Epistemology

**Questions to be thinking about:** Must mathematics be synthetic? Must it be *a priori*?

**2.2.1. How is mathematical knowledge *synthetic***?

Mathematical reasoning consists of three stages:

- *Enunciation*: the statement of the theorem to be proved
- *Ecthesis/“setting-out”*: the construction of the object only in accordance with the relevant concepts
- *Auxiliary construction*: the adding of additional information to complete the proof.

The auxiliary constructions introduce information that is not contained in the subject concept; rather it involves mental activities with the pure intuitions.

(Examples)

**2.2.2. Argument for the apriority of mathematical knowledge**

P1. We have sensory experiences that are objectively true/false with respect to space and time.

P2. If we have sensory experiences that are objectively true/false with respect to space and time, then we are committed to space and time having objective structures.

P3. **Definition:** Pure intuition = commitment to objective structure of space and time.

C1. ∴ We have pure intuitions. (from P1-P3)

P4. Knowledge about commitments is not acquired through our senses, i.e. it’s *a priori*.

C2. Knowledge about pure intuitions is *a priori*. (from P3, P4)

P5. Mathematical knowledge is knowledge about pure intuitions.

C3. ∴ Mathematical knowledge is *a priori*. (from C2, P5)

**2.2.3. What are the objective structures of space and time?**

For Kant, *Euclidean* geometry describes the objective structure of space; arithmetic describes the objective structure of time. (Often considered Kant’s major mistake; more on this tomorrow.)

**2.2.4. Why isn’t mathematical knowledge analytic?**

**2.2.4.1. The direct argument**

P1. A universal proposition is *analytic* if the predicate concept is “contained” in the subject concept.

P2. For most universal mathematical propositions, the predicate is not contained in the subject concept. (Ex. For all triangles, the sum of the interior angles is 180 degrees.)

C1. ∴ Most universal mathematical propositions are not analytic.
2.2.4.2. The existential import argument

P1. If mathematical knowledge is analytic, then mathematical knowledge involves nothing but concepts.
P2. Concepts have no existential import.
P3. Mathematical knowledge has existential import.
C1. ∴ Mathematical knowledge isn’t analytic.

2.3. Kant’s ontology

Kant is an ontological idealist: mathematical entities exist, but they are not objective; they depend on mental faculties such as our ability to apply concepts (what Kant called the understanding) and especially our ability to make mental constructions concerning space and time (what Kant called the faculty of intuition).

2.3.1. Why is Kant an ontological idealist?

Kant distinguishes between:
• The phenomenal world: everything that we can experience through our senses, and everything that we can know exists on the basis of those experiences. The structure of the phenomenal world depends on our concepts and intuitions.
• The noumenal world: the world as it is “in itself” independently of our concepts, intuitions, experiences, etc.

Kant’s ontological idealism solves his two central epistemological and metaphysical objectives:
• To refute skepticism: we have knowledge of the phenomenal world largely because the world is structured in accordance with our concepts and intuitions. We should, however, be skeptical about our knowledge of the noumenal world.
• To limit the excesses of traditional metaphysics: We can’t prove the existence of God, that we have free will, or that there is life after death, since this would involve commitments that exceed our conceptual and intuitional capacities. Traditional metaphysics errs by trying to make pronouncements on the noumenal world.

2.4. Kant’s semantics

Semantics: Truth-value antirealism has not yet made its entrance, though Kant is largely considered to be a major inspiration for it, since he’s the first to give construction a central role.

2.5. Kant on the applicability of mathematics to science

Applicability: Go back to the argument in 3.1. How do you think Kant solves the applicability problem? How does his ontological idealism also help him along?

Recall that we had an argument concerning applicability from Chapter 2 that caused trouble for idealists. Which premise of that argument would Kant reject?

3. Mill

Argued that mathematical knowledge is empirical (a posteriori).

Thoroughgoing naturalist: takes the human mind to be part of the natural, causal order.

3.1. Mill’s epistemology

Typical mathematical knowledge consists of generalizations justified via enumerative induction.
• All observed F’s are G’s.[probably]
• ∴ All F’s are G’s.

Moreover, generalizations (All F’s are G’s) add no new information: they’re “just summary records of what we have observed and what we expect to observe.” (Shapiro 2000: 93)

3.1.1. Some details

P1. The angles of all observedconstructed triangles sum (more or less) to two right angles.[probably]

C1. ∴ The angles of all triangles sum (more or less) to two right angles.

There are two (perhaps complementary) ways of understanding how Mill eliminates the “more-or-less” clause, so as to get to arrive at pure mathematical statements.
• On the first interpretation, Mill takes pure mathematics to arise from the possibility of constructing a triangle such that its angles sum exactly to two right angles. 

Problems: Unclear if such constructions are really possible. Also, as with Aristotle, the mismatch problem may lead to backdoor Platonism, which would be entirely inconsistent with Mill’s naturalism & empiricism.

• On the second interpretation, Mill takes pure mathematics to be an idealization or fiction—it’s as if triangles with angles that sum exactly to two right angles exist.

3.1.2. A problem for both interpretations

P1. If Mill is correct, then all real (empirical) propositions are justified by enumerative induction, and mathematical statements are about either possible constructions or idealizations.

P2. Neither statements about possible constructions nor those about idealizations are justified by enumerative induction; only the “more or less” claims are so justified.

C1. ∴ If Mill is correct, then mathematical statements are not real (empirical) propositions. (from P1, P2)

P3. If Mill is correct, mathematical statements are real (empirical) propositions.

C2. ∴ Mill is not correct. (from C1, P3)

3.1.3. Another problem for both interpretations: necessity

P1. If Mill is correct, mathematical statements are about either possible constructions or idealizations.

P2. Neither statements about possible constructions nor those about idealizations are necessary truths.

P3. Mathematical statements are necessary truths.

C1. ∴ If Mill is correct, there are no mathematical statements.

P4. If Mill is correct, then there are mathematical statements.

C2. ∴ Mill is not correct. (from C1, P3)

3.1.4. Mill’s response to the previous argument

Mill’s response: Math isn’t really necessary, but it appears to be so, because: (a) we’ve yet to find any empirical evidence that falsifies math, (b) we’re unable to imagine mathematical statements to be false. However, both of these are compatible with mathematics being contingent.

3.2. Mill’s ontology

Mill denies that mathematical objects are abstract (indeed, he denies the existence of any abstract objects.)

Mill’s ontology is ambiguous: geometric objects are fictions or “feigned proxies” (Shapiro 2000: 94), but numbers are (real) properties of aggregates objects.

3.3. Mill’s semantics

Truth-value realism is still the implicit norm.

3.4. Mill’s philosophy of science

Since all mathematical statements are justified by enumerative induction, and enumerative induction is a (broadly) scientific method, mathematics is highly applicable to science (see also Mill’s definition of a ‘generalization’ above).