1. Why be an intuitionist?

Intuitionism’s most (in)famous feature is its rejection of the Law of Excluded Middle (LEM).

- LEM is the claim that, for all statements \( A \), either \( A \) or not-\( A \).
- Logical and mathematical systems that include LEM are called **classical**.

Thus the intuitionist’s denial of LEM amounts to the claim that there are some statements \( A \) such that neither \( A \) nor not-\( A \).

For many, this seems like utter madness! However, intuitionists’ rejection of LEM is best seen as a consequence of their view. So first we should get some sense of why one would want to be an intuitionist. If those reasons are compelling, then denying LEM is also compelling. Moreover, intuitionists have offered several independent arguments for their position. All we need is for one of these arguments to be sound, and then LEM is on shaky ground.

1.1. The parsimony argument

P1. If a coherent mathematics does not require a metaphysical thesis \( p \), then \( p \) should not be a mathematical assumption.

P2. A coherent mathematics does not require the existence of an objective mathematical reality.

P3. That an objective mathematical reality exists is a metaphysical thesis.

C1. \( \therefore \) Mathematics should not assume the existence of an objective mathematical reality. (from P1, P2)

P4. If mathematics should not assume the existence of an objective mathematical reality, then it should not assume LEM.

C2. \( \therefore \) Mathematics should not assume LEM. (from C1, P4)

1.2. The classic idealist argument

P1. It is possible for us to know when our thoughts are correct.

P2. If it is possible for us to know when our thoughts are correct, then we know how they relate to some standard according to which our thoughts are correct.

P3. We can never know how our thoughts correspond to objective (mathematical) reality.

P4. If our thoughts correspond to objective (mathematical) reality, then our thoughts are true.

C1. \( \therefore \) Truth is not the standard according to which our thoughts are correct. (from P1-P4)

P4. If truth is not the standard according to which our thoughts are correct, then our thoughts’ justification by evidence (proofs, reasons, experience) is such a standard.

C2. \( \therefore \) Justification by evidence (proofs, reasons, experience) is the standard according to which our thoughts are correct. (from C1, P4)

P5. It is possible for a proposition \( p \) to be unjustified and for its negation, not-\( p \) to also be unjustified.

P6. Thus it is possible for neither \( p \) nor its negation, not-\( p \) to be correct.

C3. Thus (something like) LEM is not part of the standard indicating when our thoughts are correct. (from C2, P5, P6)

1.3. Dummett's argument

P1. If the classical logical system is correct, then the meaning of logical expressions (if-then, not, and, or, all, some, etc.) consists in their truth-conditions.

P2. Truth-conditions cannot be demonstrated through behavior/use of an expression.

P3. Meanings can be demonstrated behavior/use of an expression. (the manifestation requirement)

C1. \( \therefore \) The classical logical system is not correct. (from P1-P3)

P4. Either the classical logical system is correct, or the intuitionistic logical system is correct.

C2. \( \therefore \) The intuitionistic logical system is correct. (from C1, P4)

**Importantly:** unlike truth-conditions, proof-conditions can be demonstrated through behavior/use of an expression.
1.4. The generality argument

P1. Mathematics should use a maximally general logic.
P2. A logic that requires LEM is less general than a logic that does not.
C1. ∴ Mathematics should use a logic that does not require LEM. (from P1, P2)
P3. Any logic that does not require LEM is intuitionistic.
C2. ∴ Mathematics should use intuitionistic logic. (from C1, P3)

2. Inhabiting the intuitionist worldview

Classical and intuitionist systems can be distinguished by their interpretations of logical vocabulary. At the broadest level, they offer different accounts of validity:

<table>
<thead>
<tr>
<th>(classical validity) An argument is valid if and only if its conclusion is true if all of its premises are true.</th>
<th>(intuitionist validity) An argument is valid if and only if its conclusion is proven if all of its premises are proven.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(and₁) ‘A and B’ is true if and only if ‘A’ is true and ‘B’ is true.</td>
<td>(and₁) A proof of ‘A and B’ is a proof of ‘A’ and a proof of ‘B.’</td>
</tr>
<tr>
<td>(or₁) ‘A or B’ is true if and only if ‘A’ is true or ‘B’ is true.</td>
<td>(or₁) A proof of ‘A or B’ is a proof A or a proof of B.</td>
</tr>
<tr>
<td>(if₁) ‘If A then B’ is true if and only if ‘A’ is false or ‘B’ is true.</td>
<td>(if₁) A proof of ‘If A then B’ is a method of transforming any proof of ‘A’ into a proof of ‘B.’</td>
</tr>
</tbody>
</table>

But there’s more…

2.1. And, or, and if-then: the ‘unproblematic’ operators

Any proofs that rely only on these rules will be valid on both systems.

Example 1. The following argument is valid: “If Abbott is good-looking, then Khalifa is gorgeous. Abbott is good-looking. Therefore Khalifa is gorgeous.”

Classical proof:
1. An argument is valid if and only if its conclusion is true if all of its premises are true. (classical validity)
2. Suppose that both of the premises are true. (hypothesis)
3. Then the 1st premise, “if Abbott is good-looking, then Khalifa is gorgeous” is true. (2)
4. Then either “Abbott is good-looking” is false or “Khalifa is gorgeous” is true. (3, if₁)
5. The 2nd premise, “Abbott is good-looking” is true. (2)
6. Then the conclusion, “Khalifa is gorgeous,” is true (4,5).
7. So, if all of the premises are true, then the conclusion is true (2-7)
8. ∴ The argument is valid (1,7)

Intuitionistic proof:
1. An argument is valid if and only if its conclusion is proven if all of its premises are proven. (intuitionistic validity)
2. Suppose that there is a proof of both premises. (hypothesis)
3. Then there is a proof of the 1st premise, “if Abbott is good-looking, then Khalifa is gorgeous.” (2)
4. Then there is a method of transforming any proof of “Abbott is good-looking” into a proof of “Khalifa is gorgeous.” (3, if₁)
5. There is a proof of the 2nd premise, “Abbott is good-looking,” (2)
6. So there is a proof of the conclusion, “Khalifa is gorgeous.” (4,5)
7. So, if the premises are proven, the conclusion is proven (2-6)
8. ∴ So the argument is valid (1,7)
2.2. Negation: where the fun begins

Things get more interesting when considering negative statements in both systems:

<table>
<thead>
<tr>
<th>Classical</th>
<th>Intuitionist</th>
</tr>
</thead>
<tbody>
<tr>
<td>(not) ‘Not -A’ is true if and only if ‘A’ is false.</td>
<td>(not) A proof of ‘not -A’ is a proof that there is no proof of ‘A.’</td>
</tr>
</tbody>
</table>

In some cases, arguments involving negative statements will be classically valid, but intuitionistically invalid. Indeed, the differing conceptions of negation suffice to show why classicists accept LEM, and intuitionists do not:

Example 2. The statement, “Either the Seahawks will win or the Seahawks won’t win,” is necessarily true.

**Why this statement is necessary for the classicist:**

1. Suppose, for contradiction, that “Either the Seahawks will win or the Seahawks won’t win” is false. (hypothesis)
2. Then “The Seahawks will win” is false and “The Seahawks won’t win” is false (2, orc)
3. Then “The Seahawks will win” is false and “The Seahawks will win” is true (3, notc)
4. So, “Either the Seahawks will win or the Seahawks won’t win” is true. (proof by contradiction, 1-3)

**Why this statement isn’t necessary for the intuitionist:**

1. Consider the following scenario, in which the following is proven:
   a. There is no proof that “The Seahawks will win”
   b. There is no proof that “The Seahawks won’t win.” (hypothesis)
2. Then there is a proof that there is no proof that “Either the Seahawks will win or the Seahawks won’t win.” (1, or)
3. So, there is a proof that “It’s not the case that either the Seahawks will win or the Seahawks won’t win.” (2, not)

Example 3. It’s not the case that Khalifa isn’t a philosopher. So Khalifa is a philosopher.

**Classical proof of validity:**

1. An argument is valid if and only if its conclusion is true if all of its premises are true. (classical validity)
2. Suppose that the premise, “It’s not the case that Khalifa isn’t a philosopher” is true. (hypothesis)
3. Then it’s false that Khalifa isn’t a philosopher. (2, notc).
4. Then the conclusion, “Khalifa is a philosopher,” is true. (3, notc)
5. So, if all of the premises are true, then the conclusion. (2-5)
6. ∴ The argument is valid. (1,5)

**Intuitionist proof of invalidity:**

1. An argument is valid if and only if its conclusion is proven if all of its premises are proven. (intuitionist validity)
2. Consider a possible situation in which the following is proven:
   a. There is no proof that Khalifa isn’t a philosopher.
   b. There is no proof that Khalifa is a philosopher.
3. The premise, “It’s not the case that Khalifa isn’t a philosopher” is proven (2a, not)
4. The conclusion, “Khalifa is a philosopher” is unproven (2b).
5. So, it is possible that all of the premises are proven and the conclusion is unproven (3,4)
6. ∴ The argument is invalid (1,5)

3. Concepts, extensions, and other goodies

3.1. Russell’s paradox

Frege is often accused of being “naïve” set theorist, since he endorsed the following:

- Axiom of Extensionality: For any sets A and B, A = B iff \( \forall x (x \in A \iff x \in B) \)
- Axiom of Comprehension: For any condition C, there exists a set A such that \( \forall x (x \in A \iff x \) satisfies C).
1. Let \( R \) be the set of sets that are not members of themselves, i.e. \( R = \{x: \notin x\} \).
2. By 1, if \( R \notin R \), then \( R \in R \).
3. By 1, if \( R \in R \), then \( R \notin R \).
4. So, \( R \notin R \) iff \( R \in R \).
5. But this is absurd!
This implies that there is no set \( U \) which contains all sets, as \( U \) would have to contain \( R \).

### 3.2. Realists versus Intuitionists

For realists, the culprit is the Axiom of Comprehension. If it is false, then the paradox could not arise, for there need not be a set \( R \) corresponding to the condition: \( x \notin x \).

- Realists assume that the lesson of Russell's paradox is that not all collections of things are sets.

Intuitionists, by contrast, draw a very different lesson from Russell's paradox. For them, all collections are sets, and what Russell's paradox thereby entails is that \( U \) does not exist.

<table>
<thead>
<tr>
<th></th>
<th>Realist</th>
<th>Intuitionist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is ( U ) a set?</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Are all collections sets?</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>If ( U ) exists, is ( U ) a collection?</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Does ( U ) exist?</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Intuitionists also adopt a different view of concepts.

Some concepts have fixed extensions that can be determined. (Frege's mistake was to think that all concepts have fixed extensions.)

Other concepts are *indefinitely extensible*, i.e. their extensions cannot be completely determined. Given any collection of entities that falls under an indefinitely extensible concept, we can describe an entity that falls under that concept but which is not contained in that collection.

For the intuitionist, Russell's paradox reveals that the concept of set is indefinitely extensible.

This, of course, doesn't amount to a reason to accept intuitionism, but it at least shows that there's something arbitrary about realism. Moreover, intuitionists claim that they have a more principled response to Russell's paradox.

### 3.3. Quantifiers

These ideas about indefinite extensibility come to the fore when we look at classicists and intuitionists interpretation of quantifiers, such as “all,” “every,” “some,” and “there exists.”

<table>
<thead>
<tr>
<th>(all)</th>
<th>(all)</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘For all ( x ), ( A(x) )’ is true if and only if, for all objects ( d ) in the domain in question, ‘( A(d) )’ is true.</td>
<td>A proof of ‘For all ( x ), ( A(x) )’ is a procedure that, given any ( n ), produces a proof of the corresponding sentence ( A(n) ).</td>
</tr>
<tr>
<td>(some)</td>
<td>(some)</td>
</tr>
<tr>
<td>‘There is an ( x ) such that ( A(x) )’ is true if and only if for one object ( d ) in the domain in question, ‘( A(d) )’ is true.</td>
<td>A proof of ‘There is an ( x ) such that ( A(x) )’ is a construction of an item ( n ) and a proof of the corresponding sentence ‘( A(n) )’.</td>
</tr>
</tbody>
</table>

The rules will be used in the next example.

### 3.4. Math Example

One of the big complaints about intuitionism is that it “cripples the mathematician” (Shapiro 2000: 174), since less can be proven intuitionistically than classically. What would a compelling version of that argument (in standard form) look like?

In response, it's worth noting that many of the shortcomings of intuitionism don't pop up with the natural numbers\(^1\). Funkier stuff emerges when we turn to the reals.

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\(^1\) If we had a lot more time, we could explore the theorem that for every statement \( P \) that is classically provable about arithmetic, there is a statement \( P^* \) that is equivalent to \( P \) in classical logic and intuitionistically provable. Note that \( P^* \) is not always equivalent to \( P \) in intuitionistic logic, which in itself might raise worries about crippling the mathematician.
Example 5. There are infinitely many primes. In other words,
For all natural numbers \( x \), there exists a \( y \) such that \( y > x \) and \( y \) is prime.

1. Suppose, for contradiction, that the sentence, “For all natural numbers \( x \), there exists a \( y \) such that \( y > x \) and \( y \) is prime” is false. (hypothesis)
2. Then there is some natural number \( n \), such that for all \( y \), “\( y > n \) and \( y \) is prime” is false. (1, all)
3. So there is a finite number of primes: \( 2, 3, 5, …, n \). (2)
4. Let \( q = n! + 1 \)
5. Suppose that “\( q \) is prime” is true. (hypothesis)
6. Then there is a number, \( q \), that is greater than \( n \) (by 4) and is prime (by 5).
7. So there both is and is not a prime number greater than \( n \). (2, 6)
8. So “\( q \) is prime” is false. (proof by contradiction, 5-7)
9. Suppose that “\( q \) is not prime” is true. (hypothesis)
10. Then \( q \) is divisible by some prime \( p \). (assumption, but can be proven separately.)
11. \( p \) cannot be 2,3,5, … \( n \), for the division of \( q \) by these numbers would always leave a remainder of 1 (4)
12. Then there is a number, \( p \), that is greater than \( n \) (by 11) and is prime (by 10).
13. So there both is and is not a prime number greater than \( n \). (2, 12)
14. So “\( q \) is not prime” is false (proof by contradiction, 9-13)
15. So “\( q \) is prime” is true (14, notc).
16. So, “\( q \) is prime” is both true and false (8,15)
17. So, it’s true that “for all natural numbers \( x \), there exists a \( y \) such that \( y > x \) and \( y \) is prime.” (proof by contradiction, 1-19)

From the intuitionist trenches…

The moves in boldface will not yield the same result for the intuitionist. Essentially, we’d have:

(8*) There is a proof that there is no proof that “\( q \) is prime.”
(14*) There is a proof that there is no proof that “\( q \) is not prime.”

This does not lead to analogous steps at 15 and 16, since there’s no contradiction between (8*) and (14*).

However, the intuitionist can still prove the prime number theorem! For intuitionists, the concept prime is indefinitely extensible: given any collection of primes, we can find a prime number that is not contained in it.

1. Consider the following procedure:
   a. Begin with a finite number of primes: \( 2, 3, 5, …, n \).
   b. Let \( q = n! + 1 \)
   c. Then \( q \) is divisible by some prime \( p \). (assumption, but can be proven separately.)
   d. \( p \) cannot be 2,3,5, … \( n \), for the division of \( q \) by these numbers would always leave a remainder of 1. (lb)
   e. Then there exists a number, \( p \), such that \( p > n \) (by 1b) and \( p \) is prime (by 1c).
2. So there is a procedure that, given any \( n \), produces a construction of an item \( p \) and a proof of the corresponding sentence, “\( p > n \) and \( p \) is prime.” (1)
3. So there is a procedure that, given any \( n \), produces a construction of an item \( y \) and a proof of the corresponding sentence, “There exists a \( y \) such that \( y > n \) and \( y \) is prime.” (2, some)
4. There is a proof of the sentence “For all natural numbers \( x \), there exists a \( y \) such that \( y > x \) and \( y \) is prime.” (3, all)

I hope this gives you some sense of how some classical proofs can be repurposed for intuitionistic ends.
4. **Intuitionism and the four questions**

Many of the justifications for the positions below can be found in §1, above. The following serves more as a synopsis.

4.1. **Semantics**

Intuitionists are semantic antirealists (idealists) *par excellence*. They hold that mathematical statements have truth-values only relative to *us*. Specifically, statements that cannot be neither proven nor disproven do not have truth-values. Proof and disproof are mental activities. Thus, intuitionists claim that the truth-value of mathematical statements depend on our mental activities.

4.2. **Ontology**

Similarly, intuitionists are ontological idealists *par excellence*. Mathematical entities exist only relative to our mental processes of construction. (See the “some-ruile” above)

4.3. **Epistemology**

Mathematical knowledge is *synthetic a priori*.

However, intuitionists conceive of concepts and intuitions differently than Kant. (see §3.2)

4.4. **Applicability**

Different intuitionists have different views about applicability:

*Brouwer*: classical logic may be appropriate for empirical science, but pure math is different, since mathematical entities are mind-dependent. According to Shapiro (2000: 179), “this is a bold divorce between mathematics and the empirical sciences.” Agree or disagree?

*Heyting* (Shapiro 2000: 188-189):

P1. If intuitionism is true, then mathematics is about constructions and proofs.

P2. Constructions and proofs are empirical events.

C1. ‘. If intuitionism is true, then mathematics is about empirical events. (from P1, P2)